WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

501. This is true. Writing the linearity in algebra,

$$f(x) = m\,g(x) + c,$$

where m and c are constants. Differentiating both sides of this equation gives $f'(x) = m\,g'(x)$. This says that $f'(x)$ is directly proportional to $g'(x)$.

— Nota Bene —

The implication is two-way. If $f'(x) = k\,g'(x)$, then $f(x) + c_1 = k\,g(x) + c_2$. Combining the constants c_1 and c_2 gives a linear relationship.

502. Pythagoras's theorem gives the side lengths of the shaded square as $\sqrt{3^2 + 1^2} = \sqrt{10}$. Hence, the area of the shaded region is 10 (square units).

503. The interior angles of a quadrilateral add to 360° or 2π radians. So,

$$\begin{aligned}\theta &= 2\pi - \tfrac{\pi}{5} - \tfrac{2\pi}{5} - \tfrac{3\pi}{5} \\ &= \tfrac{4\pi}{5} \text{ radians.}\end{aligned}$$

504. We know that $a \propto b^2$. Squaring the relationship between b and c gives $b^2 \propto c^6$. So, $a \propto c^6$. Raising the relationship between c and d to the power 6, $c^6 \propto d^{24}$. So, $a \propto d^{24}$.

505. Solving simultaneously,

$$\begin{aligned}& x^2 + (k - x)^2 = 1 \\ \Longrightarrow\ & 2x^2 - 2kx + k^2 - 1 = 0.\end{aligned}$$

We need this quadratic to have at least one real root, so $\Delta = 4k^2 - 8(k^2 - 1) \geq 0$. This inequality has boundary equation

$$\begin{aligned}& -4k^2 + 8 = 0 \\ \Longrightarrow\ & k^2 = 2 \\ \Longrightarrow\ & k = \pm 2.\end{aligned}$$

So, we need $k \in \left[-\sqrt{2}, \sqrt{2}\right]$.

506. The remainder is zero if and only if $(x + 1)$ is a factor. So, we test $x = -1$:

$$4x^3 - 12x^2 + 18\Big|_{x=-1} = 2.$$

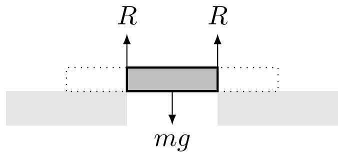
This is a non-zero remainder.

— Nota Bene —

This generalisation of the factor theorem is known as the *remainder theorem*. It states that, when $f(x)$ is divided by $(x - \alpha)$, the remainder is $f(\alpha)$.

507. In order for RR to show, the two cards with red faces must be drawn. The probability of this is $\frac{1}{3}$. Then, for success, the RG card must show R. So, the probability is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

508. Consider the central section as an object:



The middle section is 40% of the length. So, since the beam is uniform, it contains 40% of the mass, which is 8 kg. So, $2R - 8g = 0$. The vertical force exerted by each outer section is 39.2 N.

509. True. Reversing the order of the limits negates the value of a definite integral.

510. (a) Expanding the LHS,

$$\begin{aligned}(2k + 1)^2 &\equiv 4k^2 + 4k + 1 \\ &\equiv 2(2k^2 + 2k) + 1.\end{aligned}$$

(b) Since k is an integer, $2k^2 + 2k$ is an integer. Therefore $2(2k^2+2k)$ is even, so $2(2k^2+2k)+1$ is odd. Hence, the square of an odd number is always odd. QED.

511. One clock hour or five clock minutes subtends an angle $\frac{2\pi}{12} = \frac{\pi}{6}$ radians at the centre of the clock. So, the angle between the minute and hour hands at 3:30 is the angle between 3.5 clock hours and 6 clock hours, which is $2.5 \times \frac{\pi}{6} = \frac{5\pi}{12}$, as required.

512. (a) The largest angle is opposite the largest length:

$$\cos C = \frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6} = \tfrac{12}{60} = \tfrac{1}{5}.$$

(b) $A_\triangle = \frac{1}{2} \cdot 5 \cdot 6 \sin\left(\arccos \frac{1}{5}\right) = 6\sqrt{6}$.

(c) Using $A_\triangle = \frac{1}{2}\text{base} \times \text{height}$,

$$\begin{aligned}& 6\sqrt{6} = \tfrac{1}{2} \cdot 7 \cdot h \\ \Longrightarrow\ & h = \tfrac{12\sqrt{6}}{7} = 4.20 \text{ (3sf)}.\end{aligned}$$

The shortest perpendicular is at right angles to the longest side. Since this is longer than 4, the triangle must have a height of more than 4 in any orientation. Hence, it cannot be placed inside a rectangle of side length 4.

513. Each modelling term signifies that some quantity is *negligible*, where negligible means that, while the quantity in question is not exactly zero, it can be assumed to be zero, i.e. it can be neglected.

(a) "Smooth" signifies negligible friction,

(b) "Rigid" signifies negligible deformation,

(c) "Light" signifies negligible mass.

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

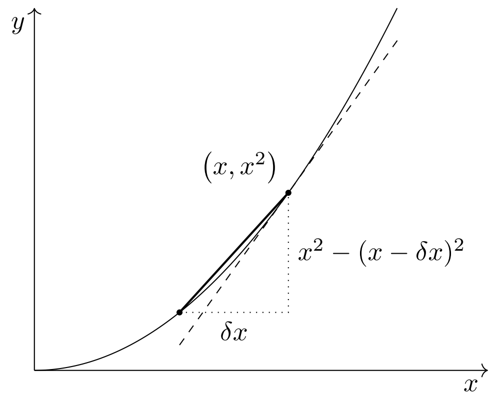
514. For $\theta \in [0, 2\pi)$, we need all values $2\theta \in [0, 4\pi)$:

$$\begin{aligned} \sin 2\theta &= \tfrac{\sqrt{3}}{2} \\ \therefore \quad 2\theta &= \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{7\pi}{3}, \tfrac{8\pi}{3} \\ \Longrightarrow \theta &= \tfrac{\pi}{6}, \tfrac{\pi}{3}, \tfrac{7\pi}{6}, \tfrac{4\pi}{3}. \end{aligned}$$

— Nota Bene —

In such a problem, the key is that the extra roots appear *at the moment the trig function is undone*, not later on. It is a common mistake to leave the generation of extra roots until the last line, which produces incorrect results.

515. (a) In this (non-standard) differentiation from first principles, the solid chord is set up to the *left* of the point at which the dashed tangent is drawn.



(b) The quantity δx is a small change in x, often called h. We will let it tend to zero, so that the gradient of the chord approaches the gradient of the tangent.

(c) We simplify and cancel a factor of δx:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{x^2 - (x - \delta x)^2}{\delta x} \\ &\equiv \lim_{\delta x \to 0} \frac{2x\delta x - (\delta x)^2}{\delta x} \\ &\equiv \lim_{\delta x \to 0} 2x - \delta x \\ &\equiv 2x, \text{ as required.} \end{aligned}$$

516. The area formula $A_\triangle = \frac{1}{2}ab\sin C$ gives $\sin\theta = 3/5$. Using the first Pythagorean identity, $\cos\theta = \pm 4/5$. The cosine rule is then

$$c^2 = 20^2 + 21^2 - 2 \cdot 20 \cdot 21 \cdot \pm\tfrac{4}{5}.$$

This tells us that $c = 13$ or $c \approx 38.9$. Since $c \in \mathbb{N}$, the third side has length 13.

517. (a) True; the constants differentiate to nothing.

(b) False; the constants 1 and 2 integrate to x and $2x$ respectively.

518. Getting cards of the same suit is more probable with replacement. Once one card of a particular suit has been picked, fewer cards remain in that suit, reducing subsequent probabilities.

519. The radius from $(0, 0)$ to (x, y) has gradient $\frac{y}{x}$. The tangent at (x, y) is perpendicular to this, so has gradient $-\frac{x}{y}$. The derivative gives the gradient of the tangent. Writing this algebraically,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

— Alternative Method —

Differentiating implicitly using the chain rule,

$$\begin{aligned} x^2 + y^2 &= 1 \\ \Longrightarrow 2x + 2y\frac{dy}{dx} &= 0 \\ \Longrightarrow \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

520. NII states that $F = ma$, with F as the resultant or unbalanced force acting on an object. If there is no unbalanced force acting, then, according to NII, $a = 0$. So, the velocity is constant. This is NI:

An object continues at constant velocity, unless an unbalanced force acts on it.

521. Rewriting the middle term,

$$\begin{aligned} x^5 - x^2\sqrt{x} - 992 &= 0 \\ \Longrightarrow x^5 - x^{\frac{5}{2}} - 992 &= 0 \\ \Longrightarrow x^{\frac{5}{2}} &= \frac{1 \pm \sqrt{1 + 4 \cdot 992}}{2} \\ \Longrightarrow x^{\frac{5}{2}} &= 32, -31. \end{aligned}$$

There are no real roots of $x^{\frac{5}{2}} = -31$, because the half index represents a square root. The solution is $x = 32^{\frac{2}{5}} = 4$.

522. Summing the side lengths, the perimeter is

$$\begin{aligned} & n\big(m^2 + k^2\big) + m\big(n^2 + k^2\big) + \big(m + n\big)\big(mn - k^2\big) \\ \equiv\ & m^2n + k^2n + mn^2 + k^2m \\ & \qquad + m^2n + mn^2 - k^2m - k^2n \\ \equiv\ & m^2n + mn^2 + m^2n + mn^2 \\ \equiv\ & 2m^2n + 2mn^2 \\ \equiv\ & 2mn(m + n), \text{ as required.} \end{aligned}$$

523. Since the parabolae rotate onto one another, their vertices must do so too. Since the rotation is by 180°, the centre of rotation must be the midpoint of the vertices. Completing the square, the vertices are $(-2, 2)$ and $(-6, 6)$. So, P is $(-4, 4)$.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

524. The formula $\theta = \pi - \frac{2\pi}{n}$ gives the interior angles:

   (a) $\frac{\pi}{2}$ radians,

   (b) $\frac{2\pi}{3}$ radians,

   (c) $\frac{5\pi}{6}$ radians.

525. If p_1 and p_2 are primes greater than 2, then they must both be odd. Since the product of two odd numbers is odd, p_1p_2+1 is even. It must be greater than 2, which means it cannot be prime. □

   — Nota Bene —

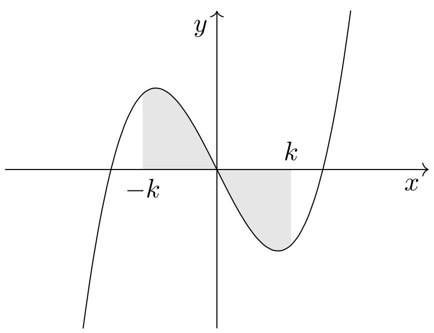
   To prove rigorously that the product of two odd numbers is odd, consider

$$\begin{aligned}&(2m+1)(2n+1)\\ \equiv\ & 4mn + 2m + 2n + 1\\ \equiv\ & 2(2mn + m + n) + 1.\end{aligned}$$

526. The value of the integral is zero:

$$\begin{aligned}&\int_{-k}^{k} x^3 - x\,dx\\ =\ & \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{-k}^{k}\\ =\ & 0.\end{aligned}$$

   Interpreting this, consider the graph $y = x^3 - x$. The integral has value zero because both the graph and the limits $x = \pm k$ have rotational symmetry around the origin.



   Hence, any signed-area contribution to the integral for positive x is cancelled out by the equivalent signed-area contribution for negative x.

527. $x \in \{1\} \implies x \in \{1, 2\}$. If x is an element of $\{1\}$, i.e. $x = 1$, then x must be an element of $\{1, 2\}$. The reverse implication is not true, with $x = 2$ as a counterexample.

528. The quantity $x^4 + y^4$ on the LHS has value zero at the origin, and increases with distance from the origin. So, we test the point $(3, 4)$:

$$x^4 + y^4 \Big|_{(3,4)} = 337 < 400.$$

   Hence, $(3, 4)$ is inside the loop.

529. Let the side lengths of the rectangles be x and y, where $x > y$. The perimeter gives $x + y = 16$, and the area gives

$$\begin{aligned}&4 \cdot \tfrac{1}{2}(x - y)y = 48\\ \implies\ & xy - y^2 = 24.\end{aligned}$$

   Substituting for x,

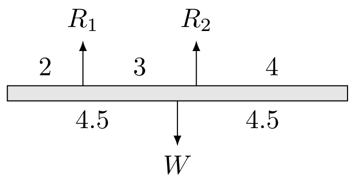
$$\begin{aligned}&(16 - y)y - y^2 = 24\\ \implies\ & y^2 - 8y + 12 = 0\\ \implies\ & y = 2, 6.\end{aligned}$$

   The first root $y = 2$ gives $x = 14$. This does not correspond to the diagram, which we are told is drawn to scale. So, $y = 6$ and $x = 10$. The area of the central square is 36.

   — Nota Bene —

   In the calculation above, we have assumed that, as shown in the diagram, the rectangles are centred on the same point. This isn't directly specified in the question. However, the perimeter and areas are unaffected if the rectangles are off-centre. So, we are justified in making the assumption.

530. (a) The force diagram is



   (b) The centre of the beam is 4.5 parts from each end. So, the two reaction forces are 2.5 parts and 0.5 parts away from the centre. Taking moments around the centre of the beam, the ratio of force magnitudes is $0.5 : 2.5$, which is $1 : 5$, as required.

531. The speed is the magnitude of the velocity. Using 3D Pythagoras,

$$\begin{aligned}&\sqrt{1 + a^2 + 4} = 3\\ \implies\ & 5 + a^2 = 9\\ \implies\ & a^2 = 4\\ \implies\ & a = \pm 2.\end{aligned}$$

532. (a) The gradient of the tangent is given by

$$\frac{dy}{dx}\bigg|_{x=p} = 2p.$$

   (b) Substituting the point (p, p^2) into $y = 2px + c$ gives $p^2 = 2p^2 + c$, so $c = -p^2$. Hence, the equation of the tangent to $y = x^2$ at (p, p^2) is $y = 2px - p^2$.

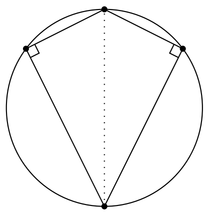
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

533. A kite has a line of reflective symmetry. Since this kite is cyclic, its opposite angles add to 180°. By symmetry, two of them must be right angles.



Hence, by the angle in a semicircle theorem, the kite's line of symmetry is be a diameter. □

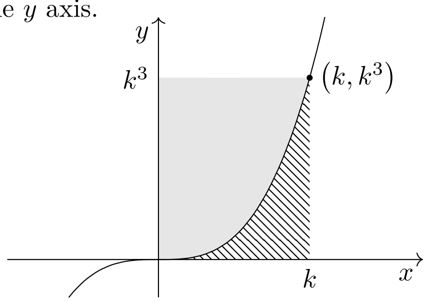
534. There is no need to split up the solution into a three, then a two, then a one. All coin tosses may be assumed to be independent, so, in total, we have a binomial distribution of six coin tosses:

$$X \sim B\big(6, \tfrac{1}{2}\big).$$

Quoting the binomial distribution formula,

$$\mathbb{P}(X = 4) = {}^6\mathrm{C}_4 \cdot \big(\tfrac{1}{2}\big)^6 = \tfrac{15}{64}.$$

535. These integrals calculate the areas of two regions defined by the point $\big(k, k^3\big)$. The integral in (a) is the hatched area between $y = x^3$ and the x axis; the integral in (b) is the solid area between $y = x^3$ and the y axis.



536. To finish in the digit zero, a number must have a factor of 10, i.e. factors of 2 and 5. In any six consecutive integers, there must be three multiples of 2 and either one or two multiples of 5. Hence, since factors of 2 and 5 must both be present, the number must end in the digit zero. QED.

537. This implication demonstrates that, <u>if</u> $\mathrm{f}(x) \equiv \mathrm{g}(x)$, <u>then</u> $0 = 0$. But the student is trying to prove <u>that</u> $\mathrm{f}(x) \equiv \mathrm{g}(x)$, which requires an implication in the opposite direction. What the student has shown is merely that there is no glaring contradiction in the statement $\mathrm{f}(x) \equiv \mathrm{g}(x)$.

— Nota Bene —

To prove an algebraic identity, there is generally no need to use implications. You begin with one side and manipulate it to reach the other: expressions are linked with identity symbols. Nevertheless, it would have been valid logic had the student shown $0 = 0 \implies \mathrm{f}(x) \equiv \mathrm{g}(x)$, since $0 = 0$ is true.

538. Differentiating,

$$\begin{aligned} y &= x^{-\frac{1}{2}} \\ \implies \frac{dy}{dx} &= -\tfrac{1}{2}x^{-\frac{3}{2}}. \end{aligned}$$

Evaluating at $x = 4$, the gradient of the tangent is $-1/16$. Taking the negative reciprocal, the gradient of the normal is 16. The point (x_1, y_1) is $(4, 1/2)$. Substituting into $y - y_1 = m(x - x_1)$,

$$\begin{aligned} y - \tfrac{1}{2} &= 16(x - 4) \\ \implies 2y &= 32x - 127, \text{ as required.} \end{aligned}$$

539. By the definition of the mean $\bar{x}$,

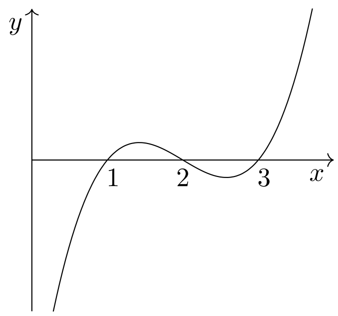
$$\sum x_i f_i = \bar{x} \sum f_i.$$

So, we solve for n in

$$\begin{aligned} &1 \cdot 1 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 17 + 5 \cdot n \\ &\qquad = 4(1 + 4 + 5 + 17 + n) \end{aligned}$$

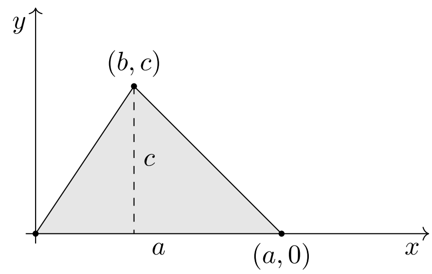
This gives $5n - 4n = 108 - 92$, so $n = 16$.

540. Consider the curve $y = (x-1)(x-2)(x-3)$. This is a positive cubic with x intercepts at $x = 1, 2, 3$:



We are looking for the set of x values for which the curve is at or below the x axis. The solution is therefore $x \in (-\infty, 1] \cup [2, 3]$.

541. Since the point $(a, 0)$ is on the x axis, the base of the triangle is on the x axis. So, the perpendicular height is in the y direction, and is given by the coordinate c.



The area is $\frac{1}{2}ac$, which does not depend on b.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

542. We need only consider the roots of the numerator, since a fraction can only be zero if its numerator is zero. And, because a^2+1 and b^2+1 are both strictly positive, the numerator has no real roots. Hence, the equation has no real roots.

543. We need to show that the new sequence w_n has a common ratio, i.e. that $\frac{w_{n+1}}{w_n}$ is constant. Using the standard nth term $u_n = ar^{n-1}$ for a GP,

$$\begin{aligned}\frac{w_{n+1}}{w_n} &= \frac{u_n + u_{n+1}}{u_{n-1}+u_n} \\ &= \frac{ru_{n-1}+ru_n}{u_{n-1}+u_n} \\ &\equiv \frac{r(u_{n-1}+u_n)}{u_{n-1}+u_n} \\ &\equiv r.\end{aligned}$$

Hence, w_n is geometric, as required.

544. (a) Substituting for y, the intersections are at $x^2 = 2x+1$, so $x_1 = 1-\sqrt{2}$ and $x_2 = 1+\sqrt{2}$.
(b) The integrand represents the height difference between the curves: since $y = 2x+1$ is above $y = x^2$, this is $2x+1-x^2$. Summing this (i.e. integrating) between $x = x_1$ and $x = x_2$ gives the area of the shaded region.
(c) Carrying out the definite integral,

$$\begin{aligned}A &= \int_{1-\sqrt{2}}^{1+\sqrt{2}} 2x+1-x^2\,dx \\ &= \left[x^2+x-\tfrac{1}{3}x^3\right]_{1-\sqrt{2}}^{1+\sqrt{2}} \\ &= \left(\tfrac{5}{3}+\tfrac{4}{3}\sqrt{2}\right)-\left(\tfrac{5}{3}-\tfrac{4}{3}\sqrt{2}\right) \\ &= \tfrac{8}{3}\sqrt{2}, \text{ as required.}\end{aligned}$$

545. The term independent of x is the constant term, which, in this case, is the middle term. Using the binomial expansion, it is

$$^4\text{C}_2 \cdot x^2 \cdot \frac{1}{x^2} = 6.$$

546. (a) The equation for intersections is $x^2-x-1=0$. This has positive discriminant, so there are two points of intersection.
(b) The equation for intersections is $x^2+1=x^2$, i.e. $1=0$. So, there are no intersections.
(c) The equation for intersections is the cubic $x^3+1=x^2$. Every cubic has at least one root, so the curves intersect.

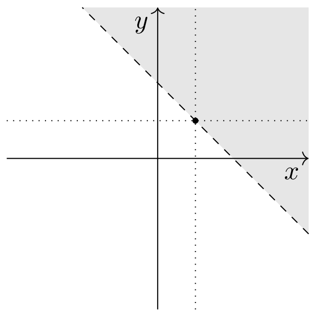
547. If an n-gon and an $(n+1)$-gon have the same perimeter, then the $(n+1)$-gon has larger area. So, since the perimeter of this $(n+1)$-gon is, in fact, larger, so must its area be.

548. By Pythagoras, the LHS and RHS are the squared distances to $(0,0)$ and (a,b). Since these are equal, the locus is the perpendicular bisector of the points $(0,0)$ and (a,b).

——— Alternative Method ———

Multiplying out, the quadratic terms x^2 and y^2 cancel, leaving $0 = -2ax + a^2 - 2by + b^2$. This is $2ax + 2by = a^2 + b^2$. For non-zero constants a, b, this is the equation of a straight line.

549. (a) The region $x+y>2$, with the point $(1,1)$ and lines $x=1$ and $y=1$ added for part (b), is



(b) Comparing shaded points with $(1,1)$, we can see that every shaded point is either to the right of $x=1$ or above $y=1$ (or both).
(c) If $x+y>2$, then one of x or y is at least 1. So, one of x^4 or y^4 is at least 1. Hence, since x^4 and y^4 are even powers, $x^4+y^4 \geq 1$. No point can therefore satisfy both inequalities.

550. Factorising as a quadratic in x^2,

$$\begin{aligned}&5x^4-6x^2+1=0 \\ \Longrightarrow\ &(5x^2-1)(x^2-1)=0 \\ \Longrightarrow\ &x^2 = 1, \tfrac{1}{5} \\ \Longrightarrow\ &x = \pm 1, \pm\tfrac{1}{\sqrt{5}}.\end{aligned}$$

551. (a) The normal distribution is symmetrical, so $-X$ is also normal: $-X \sim N(0,1)$.
(b) This distribution takes values in the set $[0,\infty)$. It is asymmetrical and cannot be normal.
(c) This is a translation of the distribution in (a): $10-X \sim N(10,1)$.
(d) As in (b), X^2 takes values in the set $[0,\infty)$. It is asymmetrical and cannot be normal.

552. For any x, the equation $\sin x = \sin y$ is satisfied by infinitely many y values. Hence, the implication is only backwards. An explicit counterexample to (a) and (c) is $x = 30°$ and $y = 150°$.
(a) False.
(b) True.
(c) False.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

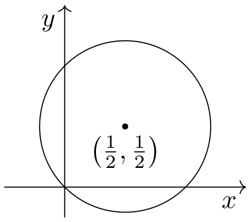
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

553. Grouping and completing the square,

$$\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{2}.$$

This is a circle, centred on $\left(\frac{1}{2},\frac{1}{2}\right)$, with radius $\frac{\sqrt{2}}{2}$. The circle passes through the origin.



554. The numerators form a difference of two squares:

$$\begin{aligned}&\frac{-b+\sqrt{b^2-4ac}}{2a}\times\frac{-b-\sqrt{b^2-4ac}}{2a}\\ \equiv\ &\frac{b^2-(b^2-4ac)}{4a^2}\\ \equiv\ &\frac{4ac}{4a^2}.\end{aligned}$$

Since $a\neq 0$, this is $\frac{c}{a}$.

555. (a) Pythagorean triples obey $a^2+b^2=c^2$. Using Euclid's formula,

$$\begin{aligned}&a^2+b^2\\ &=\left(p^2-q^2\right)^2+\left(2pq\right)^2\\ &\equiv p^4-2p^2q^2+q^4+4p^2q^2\\ &\equiv p^4+2p^2q^2+q^4\\ &\equiv \left(p^2+q^2\right)^2\\ &=c^2.\end{aligned}$$

So, any triple (a,b,c) generated by Euclid's formula is a Pythagorean triples.

(b) There are infinitely many pairs $p>q$. Each pair generates a different Pythagorean triple, which gives a different right-angled triangle with integer side lengths. Hence, there must be infinitely many right-angled triangles with integer side lengths. □

556. The ordinal formula of this GP is $u_n=2^{n-1}$. So, solving $u_n=1048576$ gives $n=21$. The sum is

$$S_{21}=\frac{1-2^{21}}{1-2}=2097151.$$

— Alternative Method —

By halving repeatedly, we know that

$$2^k=2^{k-1}+2^{k-2}+...+4+2+1+1.$$

So, the sum is

$$\begin{aligned}&1+2+4+8+...+1048576\\ &=2\times1048576-1\\ &=2097151.\end{aligned}$$

557. (a) Splitting the fraction up,

$$\begin{aligned}&y=x^{\frac{3}{2}}-2x^{\frac{1}{2}}+x^{-\frac{1}{2}}+1\\ \Longrightarrow\ &\frac{dy}{dx}=\tfrac{3}{2}x^{\frac{1}{2}}-x^{-\frac{1}{2}}-\tfrac{1}{2}x^{-\frac{3}{2}}.\end{aligned}$$

— Alternative Method —

Using the quotient rule,

$$\begin{aligned}&y=\frac{(x-1)^2}{\sqrt{x}}+1\\ \Longrightarrow\ &\frac{dy}{dx}=\frac{2(x-1)x^{\frac{1}{2}}-\frac{1}{2}(x-1)^2x^{-\frac{1}{2}}}{x}.\end{aligned}$$

(b) Solving for stationary points, we multiply by $2x^{\frac{3}{2}}$, producing a quadratic in x:

$$\begin{aligned}&\tfrac{3}{2}x^{\frac{1}{2}}-x^{-\frac{1}{2}}-\tfrac{1}{2}x^{-\frac{3}{2}}=0\\ \Longrightarrow\ &3x^2-2x-1=0\\ \Longrightarrow\ &(x-1)(3x+1)=0\\ \Longrightarrow\ &x=1,-\tfrac{1}{3}.\end{aligned}$$

The curve is not defined at $x=-\frac{1}{3}$, so the only stationary point is at $(1,1)$.

— Alternative Method —

For SPs, we require the numerator to be zero. Multiplying by $2x^{\frac{1}{2}}$,

$$\begin{aligned}&2(x-1)x^{\frac{1}{2}}-\tfrac{1}{2}(x-1)^2x^{-\frac{1}{2}}=0\\ \Longrightarrow\ &4(x-1)x+(x-1)^2=0\\ \Longrightarrow\ &(x-1)\big(4x-(x-1)\big)=0\\ \Longrightarrow\ &x=1,-\tfrac{1}{3}.\end{aligned}$$

The curve is not defined at $x=-\frac{1}{3}$, so the only stationary point is at $(1,1)$.

558. The sum of the interior angles of a decagon is 8π, so the average of them is $\frac{8}{10}\pi=\frac{4}{5}\pi$.

559. We know that $\frac{1}{2}(a+c)=b$, hence $a+c=2b$. This can be rearranged to $c-b=b-a$, which tells us that successive differences are the same. So, the sequence is an AP. □

— Alternative Method —

Assume, for a contradiction, that a,b,c are not in AP. So, $b-a\neq b-a$. This is $\frac{1}{2}(a+c)\neq b$, which contradicts the fact that b is the mean of a and c. Hence, the sequence is an AP. □

560. (a) Integrating,

$$\begin{aligned}&\int_0^1 4x^3-5x^4\,dx\\ &=\Big[x^4-x^5\Big]_0^1\\ &=\big(1-1\big)-\big(0-0\big)\\ &=0.\end{aligned}$$

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

(b) Since the definite integral above is zero, the signed area between $y = 4x^3 - 5x^4$ and the x axis is zero on $[0, 1]$. Hence, $4x^3 - 5x^4$ must take both positive and negative values over this interval. Since $4x^3 - 5x^4$ is a polynomial, this sign change guarantees a root.

561. The parabola is monic, so its leading coefficient is 1. Using the factor theorem, its equation is

$$y = (x + k)(x - 2k).$$

Substituting $(0, -8k)$ gives $-8k = -2k^2$. This has roots $k = 0$ and $k = 4$. If $k = 0$ then all intercepts are at the origin: this is not the parabola shown. So, $k = 4$ and the equation is

$$y = x^2 - 4x - 32.$$

562. Using the quadratic formula,

$$\begin{aligned} 20x^2 - 23x - 21 &= 0 \\ \Longrightarrow x &= -\tfrac{3}{5}, \tfrac{7}{4}. \end{aligned}$$

So, by the factor theorem,

$$20x^2 - 23x - 21 \equiv (5x + 3)(4x - 7).$$

The coefficients of x are 5 and 4. On the RHS of the original identity, they are a and $a - 1$. So, $a = 5$. Then $b = 3$ and $c = -7$.

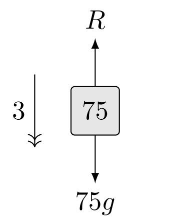
563. A translation by $2\mathbf{i}$ requires a replacement of x by $x - 2$, and a translation by $3\mathbf{j}$ requires an addition of 3 (equivalent to a replacement of y by $y - 3$). This gives

$$\begin{aligned} y &= (x - 2)^2 + (x - 2) + 3 \\ &\equiv x^2 - 3x + 5. \end{aligned}$$

564. The first statement is true, as $y = 0 \iff y^2 = 0$. The second is not, since the backwards implication does not hold: $\mathrm{f}(a) = -1$ is a counterexample. The correct implication is

$$\mathrm{f}(a) = 1 \implies \big(\mathrm{f}(a)\big)^2 = 1.$$

565. The force diagram for the fridge is



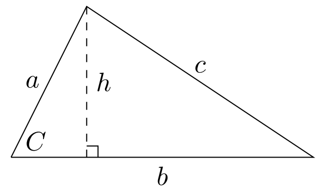
The equation of motion is $75g - R = 75 \cdot 3$, so $R = 510$. This is the reaction exerted *by* the lift *on* the fridge, which is the NIII pair of the reaction exerted *on* the lift *by* the fridge. This latter force has magnitude 510 N, acting vertically downwards.

— Nota Bene —

In an accelerating lift, weight and reaction are not equal. This is a counterexample to the common misconception that weight and contact force are "action" and "reaction". This is why Newton's original formulation of the third law, in which he famously juxtaposed those terms, is no longer helpful for understanding. Language has changed in the intervening centuries.

In fact, each force has its own Newton pair. There are two weights/gravitational forces: one acts downwards on the fridge, the other acts upwards on the Earth. And there are two contact/reaction forces: one acts upwards on the fridge, the other acts downwards on the lift.

566. In triangle ABC, drop a perpendicular from vertex B to side AC:



This perpendicular has length $h = a \sin C$. Hence, the triangle has area $A_\triangle = \frac{1}{2}bh = \frac{1}{2}ab \sin C$. QED.

567. The input transformation $x \mapsto 2x$ does not affect the ranges, which are sets of output values. By definition, the sin and cos functions only generate outputs which are coordinates on the unit circle. Dividing one by the other for tan can produce any real number.

(a) $[-1, 1]$,

(b) $[-1, 1]$,

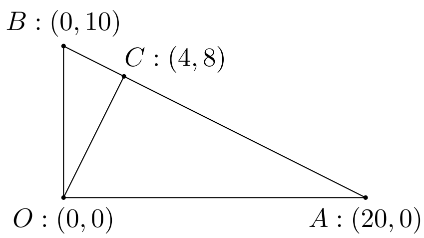
(c) $\mathbb{R}$.

568. (a) Labelling the points O, A, B and C,

$$\overrightarrow{AB} = \begin{pmatrix} -20 \\ 10 \end{pmatrix} = \tfrac{5}{4}\begin{pmatrix} -16 \\ 8 \end{pmatrix} = \tfrac{5}{4}\overrightarrow{BC}.$$

Since $\overrightarrow{AB} = k\overrightarrow{BC}$, points A, B, C are collinear.

(b) The scenario is



The two vectors $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} -20 \\ 10 \end{pmatrix}$ are perpendicular, since $8/4 \times 10/{-20} = -1$. So, OAC and OBC are right-angled triangles. $\triangle AOB$ is also right-angled, proving the result.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

569. With $a = 1$, the LHS is

$$\begin{aligned}&(-1)^{-1}\binom{2}{0}^3 + (-1)^0\binom{2}{1}^3 + (-1)^1\binom{2}{2}^3\\ &= -1^3 + 2^3 - 1^3\\ &= 6.\end{aligned}$$

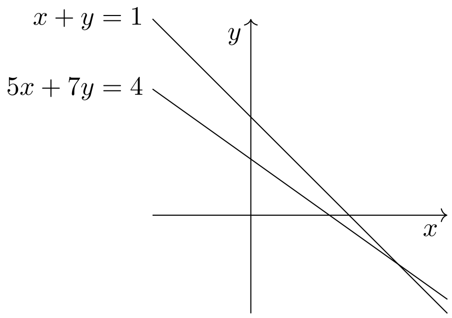
The RHS is $\dfrac{3!}{(1!)^3} = 6$, as required.

570. (a) Mass is a conserved, permanent quantity in the Newtonian model, so cannot drop to zero.

(b) The gravitational force exerted on an object, which is also known as the object's weight, drops to zero (or more precisely is negligible) in deep space.

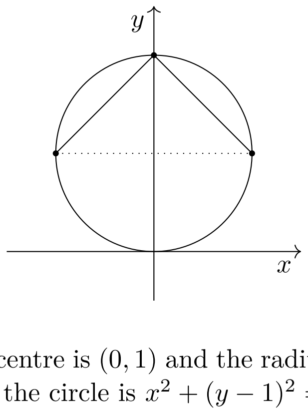
(c) Reaction force can drop to zero in the presence of a gravitational field, when a human being has e.g. jumped from a diving board.

571. In the positive quadrant of a set of (x, y) axes, the boundary line $5x + 7y = 4$ is below and to the left of the boundary line $x + y = 1$:



Hence, if $x + y > 1$, then (x, y) is above and to the right of $x + y = 1$, therefore it is above and to the right of $5x + 7y = 4$. This proves the implication.

572. Since the line segments are chords, their endpoints, which have $s \in \{0, 1\}$ and $t \in \{0, 1\}$, must lie on the circle. These points are $(0, 2)$, $(1, 1)$ and $(-1, 1)$, $(0, 2)$. Since the chords are at right-angles to each other, $(1, 1)$ and $(-1, 1)$ must be endpoints of a diameter.



Hence, the centre is $(0, 1)$ and the radius is 1. The equation of the circle is $x^2 + (y - 1)^2 = 1$.

573. Algebraically, the iteration is $A_{n+1} = 3 + 2A_n$.

① Subtracting A_n from the iterative definition gives the term-to-term difference as

$$A_{n+1} - A_n = 3 + A_n.$$

We are told that the sequence is increasing, so the RHS cannot be constant. Likewise the LHS, so the sequence cannot be arithmetic.

② Dividing the iterative definition by A_n gives the term-to-term ratio as

$$\frac{A_{n+1}}{A_n} = \frac{3}{A_n} + 1.$$

Since A_n is non-constant, the term-to-term ratio must vary. Hence, the sequence cannot be geometric.

574. Triangle AXC is congruent to triangle ABC. So, the length of the dashed perpendicular is $\frac{1}{2}|AC|$, which is $\sqrt{2}/2$.

575. The equation for intersections is

$$\begin{aligned}&x^2 + 1 = -4x - 3x^2\\ \Longrightarrow\ &4x^2 + 4x + 1 = 0\\ \Longrightarrow\ &(2x + 1)^2 = 0.\end{aligned}$$

This has a double root at $x = -\frac{1}{2}$. Hence, the two curves are tangent at this point.

576. (a) The input transformation stretches by factor 2 in the x direction. So, the period of the new function is 6.

(b) The output transformations have no effect on the period, which remains at 5.

(c) The sum has period $\text{lcm}(3, 5) = 15$.

577. The curves intersect at $x = \pm 2$. The area is given by the integral of the difference. The line $y = 4$ is above $y = x^2$, so the integrand is $4 - x^2$:

$$\begin{aligned}A &= \int_{-2}^{2} 4 - x^2\, dx\\ &= \left[4x - \tfrac{1}{3}x^3\right]_{-2}^{2}\\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)\\ &= \tfrac{32}{3}.\end{aligned}$$

578. There are eight outcomes in the possibility space, of which two, HHH and TTT, have all three coins showing the same. Hence,

$$\begin{aligned}\mathbb{P}(S) &= \tfrac{2}{8}\\ \mathbb{P}(S') &= \tfrac{6}{8}.\end{aligned}$$

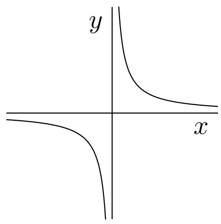
This gives $\mathbb{P}(S') = 3\,\mathbb{P}(S)$, as required.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

579. There is only one real cube root of a number. So, the curve $x^3y^3 = 1$ is identical to $xy = 1$. This is the standard reciprocal graph $y = 1/x$:



580. (a) $F = ma$ horizontally and vertically gives

$$R \sin\theta = 80,$$
$$R \cos\theta = 60.$$

(b) Squaring and adding the results from part (a),

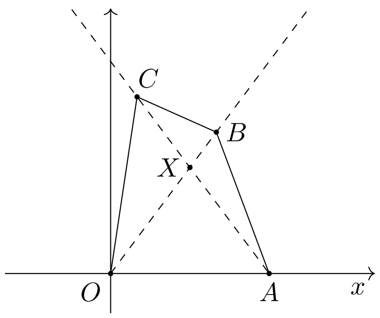
$$R^2(\sin^2\theta + \cos^2\theta) = 80^2 + 60^2.$$

Using the first Pythagorean identity,

$$R^2 = 100$$
$$\therefore \quad R = 10.$$

Dividing the equations gives $\tan\theta = \frac{4}{3}$. So, $\theta = \arctan\frac{4}{3} = 53.1°$ (1dp).

581. Call the given vertices O, A and the intersection X. Since O, A and X are not collinear, the remaining two vertices B, C must be on the lines AX and OX, extended beyond X.



- Since B and C could be arbitrarily far away, there is no upper bound on the perimeter P.
- To make the perimeter as small as possible, B and C could be arbitrarily close to (but not exactly at) X. The lower bound, therefore, is the perimeter of $\triangle OAX$. This perimeter is 16. This value cannot be attained, so we exclude it from the interval.

The perimeter satisfies $P \in (16, \infty)$.

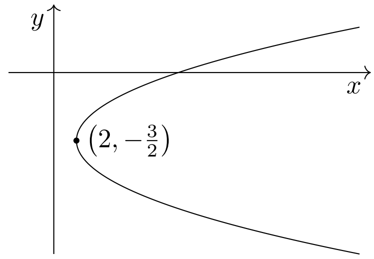
582. Taking out a factor of $(x - 1)$,

$$(x-1)\big((x-2)+(x-3)\big) = 0$$
$$\Longrightarrow (x-1)(2x-5) = 0$$
$$\Longrightarrow x = 1, \tfrac{5}{2}.$$

583. (a) $x = 4\left(y + \frac{3}{2}\right)^2 + 2$.

(b) The vertex is at $\left(2, -\frac{3}{2}\right)$.

(c) The parabola is positive, so looks like:



584. To translate by $5\mathbf{i}$, we add 5 to the value of the x coordinate. This gives $x = t + 9$, $y = 3t$, for $t \in \mathbb{R}$.

——— Alternative Method ———

To translate by $5\mathbf{i}$, we replace x by $x - 5$. This gives $x - 5 = t + 4$, and doesn't affect y, so the image is $x = t + 9$, $y = 3t$, for $t \in \mathbb{R}$.

585. Factorising numerator and denominator,

$$\frac{p^{\frac{5}{2}} - p^{\frac{3}{2}}}{p^{\frac{3}{2}} - p^{\frac{1}{2}}}$$
$$\equiv \frac{p^{\frac{3}{2}}(p-1)}{p^{\frac{1}{2}}(p-1)}$$
$$\equiv p, \text{ where } p \neq 0, 1.$$

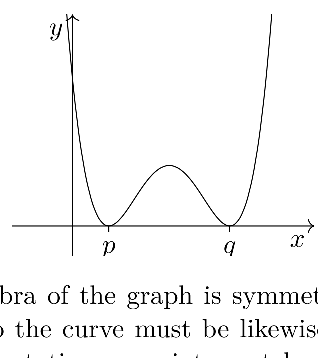
586. We know that $C = 2\pi r$. Differentiating both sides with respect to t gives

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}.$$

So, the rate of change of the circumference is 2π times greater than the rate of change of the radius. This gives 10π cm/s.

587. (a) Each factor $(x - p)$ and $(x - q)$ is squared. These squared factors correspond to double roots, which are points at which the curve is parabolically tangent to the x axis. Hence $x = p$ and $x = q$ must both be stationary points.

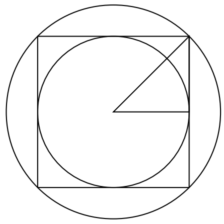
(b) Sketch of $y = (x - p)^2(x - q)^2$:



(c) The algebra of the graph is symmetrical in p and q, so the curve must be likewise. Hence, the third stationary point must be at the x-midpoint of p and q, so $x = \frac{1}{2}(p + q)$.

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

588. The scenario is:



The triangle in the diagram has side lengths in the ratio $1 : \sqrt{2}$. This is the ratio of the radii of the circle, so the ratio of areas is $1 : 2$, as required.

589. Using 3D Pythagoras, the magnitude is

$$\begin{aligned}|\mathbf{r}| &= \sqrt{\left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2} \\ &= 1.\end{aligned}$$

So, $\mathbf{r}$ is a unit vector.

590. Multiplying both sides by $x(1-x)(1+x)$,

$$1 - x^2 \equiv Px(1+x) + Qx(1-x).$$

Equating coefficients,

$$\begin{aligned} x^2 &: -1 = P - Q \\ x^1 &: 0 = P + Q \\ x^0 &: 1 = 0. \end{aligned}$$

Since the constant terms do not match, such an identity can never hold.

591. Using the fact that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$:

(a) $\mathbb{Z} \cup \mathbb{R} = \mathbb{R}$,

(b) $\mathbb{Z} \cap \mathbb{N} = \mathbb{N}$,

(c) $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$.

— Nota Bene —

The sets are:

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers (*Zahl* is German for number)
$\mathbb{Q}$	Rationals (Quotient is a word for fraction)
$\mathbb{R}$	Real numbers.

592. Multiplying out,

$$\begin{aligned}\mathrm{f}(x) &= x^2\big(1 + x^n\big) \\ &\equiv x^2 + x^{n+2}\end{aligned}$$

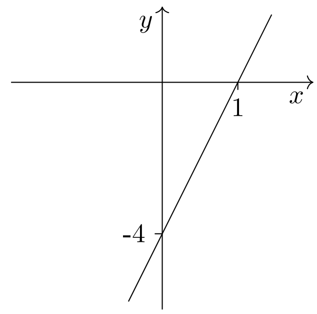
The derivative is

$$\begin{aligned}\mathrm{f}'(x) &= 2x + (n+2)x^{n+1} \\ &\equiv x\big(2 + (n+2)x^n\big).\end{aligned}$$

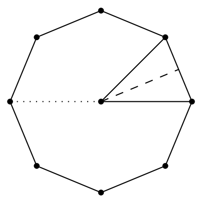
Hence, $a = 2$, $n = 3$ and $b = 5$.

593. Every AP is either constant, or diverges to $\pm\infty$. If the AP is constant, then so will its reciprocals be. If the AP diverges to $\pm\infty$, then its reciprocals must tend to zero. QED.

594. The integral between 0 and 1 has value -2. The domain $[0,1]$ has width 1, so the average value of $\mathrm{f}(x)$ on $[0,1]$ must be -2. At one endpoint of the domain, we know that $\mathrm{f}(1) = 0$. At the other end, therefore, we need $\mathrm{f}(0) = -4$:



595. Each triangular sector of the octagon is isosceles and subtends an angle $\frac{2\pi}{8} = \frac{\pi}{4}$ at the centre. So, we can split one into two right-angled triangles, each subtending $\frac{\pi}{8}$.



The opposite is half an edge, so has length $\frac{1}{2}$. The hypotenuse, therefore, has length

$$\frac{1}{2\sin\frac{\pi}{8}} = \frac{1}{\sqrt{2-\sqrt{2}}}.$$

The diameter is then twice the hypotenuse:

$$2 \times \frac{1}{\sqrt{2-\sqrt{2}}} = \frac{2}{\sqrt{2-\sqrt{2}}}.$$

596. Two square numbers are given by a^2 and b^2, where $a, b \in \mathbb{Z}$. Their product is $a^2b^2 \equiv (ab)^2$. Since $ab \in \mathbb{Z}$, this is a square number. □

597. Writing the powers over base 2,

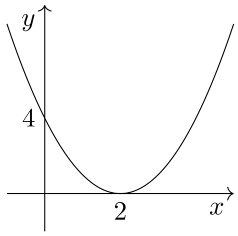
$$\begin{aligned} & 2^x \times 4^{x-1} \times 8^{x-2} = 1 \\ \Longrightarrow\ & 2^x \times 2^{2(x-1)} \times 2^{3(x-2)} = 1 \\ \Longrightarrow\ & 2^{6x-8} = 1 \\ \Longrightarrow\ & 6x - 8 = 0 \\ \Longrightarrow\ & x = \tfrac{4}{3}. \end{aligned}$$

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

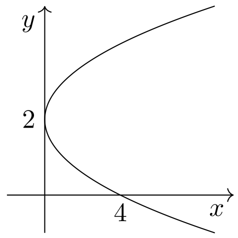
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

598. (a) The information $g(2) = g'(2) = 0$ tells us that the vertex of the first parabola is at $(2, 0)$. It is monic, so its leading coefficient is 1, giving the y intercept as $y = 4$:



(b) The second parabola is then a reflection of the first in $y = x$:



599. To add vectors is to connect them tip-to-tail in the manner described in this question. And the sum of three vectors, if they maintain equilibrium, must be zero. Hence, if the three vectors are connected in this manner, the distance from the overall tail to the overall tip must be zero. This is the same as saying that the vectors must form a closed triangle.

600. (a) $45°$ per hour is $\frac{\pi}{4}$ radians per hour, which is $\frac{\pi}{4} \div 3600 = 2.18 \times 10^{-4}$ rad/s (3sf).

(b) 1.2×10^{-4} rpm is $2.4\pi \times 10^{-4}$ rad/m, which is $2.4\pi \times 10^{-4} \div 60 = 1.26 \times 10^{-5}$ rad/s (3sf).

——— End of 6th Hundred ———

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK